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Optimal chattering modes in the problem of the control of a Timoshenko beam☆

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Abstract

The linear problem of the control in a plane of the motion of a Timoshenko beam, one end of which is clamped to a rotating disc is considered. The angular acceleration of the disc serves as the control. It is proved that, in the problem of the quenching of the first mode, the optimal control has an infinite number of switchings in a finite time interval (a chattering control). The construction of a suboptimal control with a finite number of switchings is described.

In the theory of the optimal control of the vibrations which arise in mechanical systems, a model of an Euler–Bernoulli beam has been the object of numerous investigations. The Timoshenko beam theory is a development of the Euler–Bernoulli system, which takes into account the rotational inertia and the shearing deformation of the section which occurs during the vibrations. The basic assumption in the Timoshenko beam model is that planar sections, normal to the beam axis prior to deformation, also remain planar after the deformation of the beam, rotating as a rigid whole through an angle $\xi(x, t)$ during which the sections do not necessarily remain normal to the beam axis.[1,2] The model of a homogeneous Timoshenko beam is described by a system of second-order partial differential equations

$$\rho P w_{tt} - kPGw_{xx} + kPG\xi = 0, \quad \rho I\xi_{tt} - EI\xi_{xx} + kPG\xi - kPGw_x = 0$$

where ρ is the density of the mass of the beam, E is Young's modulus of elasticity, G is the shear modulus, I is the moment of inertia, P is the cross-section area, k is the Timoshenko shear coefficient, and $w(x, t)$ and $\xi(x, t)$ are the displacement of the beam in a direction perpendicular to the axis of the beam at the position of rest and the angular displacement of the cross-section of the beam at the instant t at a point x respectively.

In investigating the problem of controlling a slowly rotating Timoshenko beam, the controllability of the beam from a specified position into a position of rest within a quite long time has been proved in Refs. 3,4. A control was constructed in Ref. 5 which stabilizes the whole system at the position of rest. The conditions for exact controllability were obtained in Ref. 6 and reachability sets have been described.

The problem of minimizing the mean square deviation of a Timoshenko beam from the equilibrium position is considered below. It is proved that, in the problem of the optimal quenching of the first mode, the control has an infinite number of switchings in a finite time interval.

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1. Formulation of the problem

We will consider a model of a Timoshenko beam under the assumption that one end of the beam is clamped to a disk of radius r. It is assumed that the motion of the beam is controlled by the acceleration of the disc. By using rescaling, the equations of a Timoshenko beam can be reduced to a system with a single real parameter $\gamma > 0$[3]

$$\begin{aligned} w_{tt}(x,t) - \frac{1}{\gamma}w_{xx}(x,t) + \frac{1}{\gamma}\xi_x(x,t) &= -\ddot{\theta}(t)(r+x) \\ \xi_{tt}(x,t) - \xi_{xx}(x,t) + \frac{1}{\gamma}\xi(x,t) - \frac{1}{\gamma}w_x(x,t) &= -\ddot{\theta}(t) \end{aligned} \tag{1.1}$$

where $\theta(t)$ is the angle of rotation of the disc at the instant of time t. The boundary conditions

$$w(0,t) = \xi(0,t) = 0, \quad w_x(l,t) - \xi(l,t) = 0, \quad \xi_x(l,t) = 0 \tag{1.2}$$

are specified, where l is the length of the beam. They signify that one end of the beam ($x=0$) is clamped and the other ($x=l$) is free. We now define the initial conditions

$$w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ \xi(x,0) = \xi_0(x),\ \xi_t(x,0) = \xi_1(x),\ x \in [0,l] \tag{1.3}$$

We shall henceforth assume that $l=1$ and $\gamma=1$ and consider the problem of minimizing the deviation of the beam from the equilibrium position in the sense of the following functional

$$\int_0^\infty \|\boldsymbol{\omega}(x,t)\|_H^2 dt \tag{1.4}$$

where $\boldsymbol{\omega}(x,t) = \begin{pmatrix} w(x,t) \\ \xi(x,t) \end{pmatrix}$ and H is the Hilbert space $L^2([0,1],\mathbb{C}^2)$, that is, a linear space of the vector functions $\mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ which are such that

$$\int_0^\infty (|y_1(x)|^2 + |y_2(x)|^2) dx < \infty$$

with a scalar product

$$\langle \mathbf{y}(x), \mathbf{z}(x) \rangle_H = \int_0^1 (y_1(x)\overline{z_1(x)} + y_2(x)\overline{z_2(x)}) dx$$

We now consider a subset of the Hilbert space H

$$D = \{\mathbf{y}(x) \in H | y_1(0) = y_2(0) = 0, \quad y_1'(1) - y_2(1) = 0, \quad y_2'(1) = 0\} \tag{1.5}$$

and define a linear differential operator A: $D \to H$ by the formula

$$A\mathbf{y} = \begin{pmatrix} -y_1'' + y_2' \\ -y_1' - y_2'' + y_2 \end{pmatrix}$$

Here, a prime denotes differentiation with respect to x.

The problem of the optimal control of the beam can then be written in the following form: it is required to minimize the functional (1.4) along the trajectories of the control system

$$\frac{\partial^2}{\partial t^2}\boldsymbol{\omega}(\cdot,t) + A\boldsymbol{\omega}(\cdot,t) = u(t)\mathbf{g}(\cdot), \quad t > 0; \tag{1.6}$$

where $\mathbf{g}(x) = \begin{pmatrix} -(r+x) \\ -1 \end{pmatrix}$ and $u(t) = \ddot{\theta}(t)$ is a scalar control, which we shall assume satisfies the constraint $-1 \le u \le 1$.

2. The eigenvalues and eigenfunctions of the differential operator

It has been proved in Refs. 3,4 that A is a strictly positive, self-adjoint operator which possesses a complete orthonormal system of eigenfunctions $\mathbf{h}_j(x)(j \in \mathbb{N})$ and a corresponding system of eigenvalues $\{\lambda_j \in \mathbb{R}\}$ which are such that $1 < \lambda_j \uparrow \infty$ as $j \to \infty$.

Theorem 1. *The eigenvalues λ_j solve the equation*

$$\frac{8\lambda^{3/2}}{\lambda - 1}\left(1 + \cos\sqrt{\lambda - \sqrt{\lambda}}\cos\sqrt{\lambda + \sqrt{\lambda}} - \sqrt{\frac{\lambda}{\lambda - 1}}\sin\sqrt{\lambda - \sqrt{\lambda}}\sin\sqrt{\lambda + \sqrt{\lambda}}\right) = 0 \tag{2.1}$$

Proof. Suppose λ is an eigenvalue of the operator A, and $\mathbf{h}(x)$ is the corresponding eigenfunction, that is,

$$A\mathbf{h}(x) = \lambda\mathbf{h}(x), \quad \mathbf{h}(x) \not\equiv 0 \tag{2.2}$$

We now put

$$\delta^{\pm} = \sqrt{\lambda \pm \sqrt{\lambda}}, \quad \mu^{\pm} = \sqrt{\lambda} \pm 1 \tag{2.3}$$

The general solution of Eq. (2.2) has the form

$$\mathbf{h}(x) = D_1^+\boldsymbol{\chi}_1^+e^{-i\delta^+} + D_1^-\boldsymbol{\chi}_1^-e^{i\delta^+} + D_2^+\boldsymbol{\chi}_2^+e^{-i\delta^-} + D_2^-\boldsymbol{\chi}_2^-e^{i\delta^-}$$

where

$$\boldsymbol{\chi}_1^{\pm} = \begin{pmatrix} -1 \\ \dfrac{\pm i\delta^+}{\mu^+} \end{pmatrix}, \quad \boldsymbol{\chi}_2^{\pm} = \begin{pmatrix} -1 \\ \dfrac{\pm i\delta^-}{\mu^-} \end{pmatrix}$$

and $D_1^{\pm}$ and $D_2^{\pm}$ are certain complex constants. Since the function $\mathbf{h}(x)$ satisfies conditions (1.5), the constants $D_1^{\pm}$ and $D_2^{\pm}$ are determined from the homogeneous system of algebraic equations

$$\begin{gathered}
-D_1^+ - D_1^- + D_2^+ + D_2^- = 0 \\
\frac{D_1^+\delta^+}{\mu^+} - \frac{D_1^-\delta^+}{\mu^+} + \frac{D_2^+\delta^-}{\mu^-} - \frac{D_2^-\delta^-}{\mu^-} = 0 \\
D_1^+\sqrt{\lambda}e^{-i\delta^+} + D_1^-\sqrt{\lambda}e^{i\delta^+} + D_2^+\sqrt{\lambda}e^{-i\delta^-} + D_2^-\sqrt{\lambda}e^{i\delta^-} = 0 \\
\frac{D_1^+\delta^+\sqrt{\lambda}e^{-i\delta^+}}{\mu^+} - \frac{D_1^-\delta^+\sqrt{\lambda}e^{i\delta^+}}{\mu^+} - \frac{D_2^+\delta^-\sqrt{\lambda}e^{-i\delta^-}}{\mu^-} + \frac{D_2^-\delta^-\sqrt{\lambda}e^{i\delta^-}}{\mu^-} = 0
\end{gathered} \tag{2.4}$$

A non-zero solution of system (2.4) exists if and only if the determinant of this system is equal to zero, which is equivalent to Eq. (2.1).

Since $\lambda_j > 1$, to calculate the eigenvalues it is sufficient to consider the equation which corresponds to the expression in brackets in (2.1) being equal to zero:

$$1 + \cos\delta^-\cos\delta^+ - \tilde{\lambda}\sin\delta^-\sin\delta^+ = 0; \quad \tilde{\lambda} = \sqrt{\frac{\lambda}{\lambda - 1}} \tag{2.5}$$

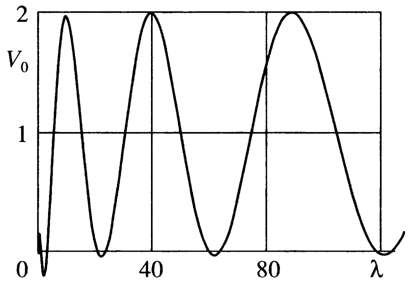

Fig. 1.

or what is equivalent to this:

$$\frac{2}{1+\tilde{\lambda}} + \cos(\delta^- + \delta^+) + \frac{1-\tilde{\lambda}}{1+\tilde{\lambda}}\cos(\delta^- - \delta^+) = 0 \tag{2.6}$$

We recall that $\delta^{\pm}$ depends on λ by virtue of expressions (2.3). The left-hand side of Eq. (2.6) is denoted by $V_0(\lambda)$ and a graph of the function $V_0(\lambda)$ is shown in Fig. 1.

We will now study the asymptotic form of the eigenvalues of the operator A when $j \to \infty$. We put $\sigma = \sqrt{\lambda}$ and define the function

$$\Phi(\sigma) = V_0(\sigma^2) = \frac{2\sqrt{\sigma^2-1}}{\sqrt{\sigma^2-1}+\sigma} + \cos(\sqrt{\sigma^2-\sigma} + \sqrt{\sigma^2+\sigma}) +$$

$$+ \frac{\sqrt{\sigma^2-1}-\sigma}{\sqrt{\sigma^2-1}+\sigma}\cos(\sqrt{\sigma^2-\sigma} - \sqrt{\sigma^2+\sigma})$$

When $\sigma \to \infty$, the function $\Phi(\sigma)$ is asymptotically equivalent to the following function:

$$\Phi_0(\sigma) = 1 + \cos(2\sigma) + \frac{1}{4\sigma}\sin(2\sigma) - \frac{1}{4\sigma^2}\left(1 + \cos 1 + \frac{\cos(2\sigma)}{8}\right)$$

Lemma 1. *Constants $N_0 > 0$, $B_1 > 0$ and $B_2 > 0$ exist such that, for all $\sigma \geq N_0$,*

$$|\Phi(\sigma) - \Phi_0(\sigma)| \leq B_1/\sigma^3, \quad |\Phi'(\sigma) - \Phi_0'(\sigma)| \leq B_2/\sigma^3$$

The lemma is proved by direct calculations.

Lemma 2. *A constant $N_1 > 0$ exists such that, in the domain $\sigma \geq N_1$, the roots of the equation $\Phi_0(\sigma) = 0$ have the form*

$$\sigma_k^{\pm} = \zeta_k + \frac{1}{8\zeta_k}\left(1 \pm 4\cos\frac{1}{2}\right) + O\left(\frac{1}{\zeta_k^2}\right) \tag{2.7}$$

where $\zeta_k = \pi(2k-1)/2$, $k \in \mathbb{N}$ are the solutions of the equation $1 + \cos(2\sigma) = 0$.

Proof. Consider the Taylor expansion of the function $\Phi_0(\sigma)$ in the neighbourhood of the point ζ_k

$$\Phi_0(\sigma) = -\frac{7}{32\zeta_k^2} - \frac{\cos 1}{4\zeta_k^2} - \frac{1}{2\zeta_k}(\sigma - \zeta_k) + \left(2 + \frac{7}{13\zeta_k^2}\right)(\sigma - \zeta_k)^2 + O\left(\frac{1}{\zeta_k^3}\right)$$

Solving the equation $\Phi_0(\sigma) = 0$, we obtain the equality (2.7).

Hence, two different roots of the equation $\Phi_0(\sigma) = 0$ are found for sufficiently large σ in a small neighbourhood of the point ζ_k. Note that the distance between them is of the order of $1/\zeta_k$.

Lemma 3. *Positive constants m^+ and m^- exist such that*

$$\lim_{k\to+\infty}(\zeta_k\Phi_0'(\sigma_k^\pm)) = \pm m^\pm$$

Proof of Lemma 3 *is easily obtained from the definition of the function $\Phi_0(\sigma)$ and* Lemma 2.

It follows from Lemmas 1–3 and the theorem on an inverse function, applied to $\Phi(\sigma)$ at the point $\sigma_k^\pm$, that a pair of roots $\kappa_k^\pm$ of the equation $\Phi(\sigma)=0$, which has no other roots, corresponds to each pair of roots $\sigma_k^\pm$ of the equation $\Phi_0(\sigma)=0$.

Lemma 4. *As $k\to+\infty$, the following estimate is true for the roots of the function $\Phi(\sigma)$*

$$\kappa_k^\pm = \sigma_k^\pm + O\left(\frac{1}{\zeta_k^2}\right)$$

The following estimate for the eigenvalues of the operator A can be obtained from Lemma 4.

Theorem 2. *Two similar eigenvalues λ_k^- and λ_k^+ of the operator A correspond to each sufficiently large value of k such that*

$$\lambda_k^- < (\pi(2k-1)/2)^2 < \lambda_k^+,\quad \lambda_k^+ - \lambda_k^- = 2\cos\frac{1}{2} + O\left(\frac{1}{\zeta_k}\right) \tag{2.8}$$

Proof. It follows from the definition of the function $\Phi(\sigma)$ that an eigenvalue λ of the operator A corresponds to each root of the equation $\Phi(\sigma)=0$. We put $\lambda_k^\pm = (\kappa_k^\pm)^2$.

On applying Lemmas 2 and 4, we obtain

$$\lambda_k^\pm = \left(\zeta_k + \frac{1\pm 4\cos\frac{1}{2}}{8\zeta_k} + O\left(\frac{1}{\zeta_k^2}\right)\right)^2 = \zeta_k^2 + \frac{1\pm 4\cos\frac{1}{2}}{4} + O\left(\frac{1}{\zeta_k}\right)$$

from which relation (2.8) follows.

Remark. The distance between the different pairs of similar eigenvalues of the operator A increases linearly with k.

The first thirty two eigenvalues of the operator A are shown in Table 1.

Remark. An operator $\tilde{A}$ was studied in Ref. 4 which is somewhat different from the operator A. However, the spectrum of $\tilde{A}$ is identical to the spectrum obtained in the case of the operator A. The operators A and $\tilde{A}$ obviously reduce to one another.

Table 1

k	1	2	3	4	5	6	7	8
ζ_k^2	2.467	22.207	61.685	120.903	199.859	298.556	416.991	555.165
λ_k^-	1.623	21.568	61.054	120.273	199.231	297.927	416.363	554.537
λ_k^+	3.587	23.333	62.812	122.030	200.987	299.683	418.118	556.293
$\lambda_k^+ - \lambda_k^-$	1.964	1.765	1.759	1.757	1.757	1.756	1.756	1.756

k	9	10	11	12	13	14	15	16
ζ_k^2	713.079	890.732	1088.124	1305.255	1542.126	1798.108	2075.084	2371.172
λ_k^-	712.451	890.104	1087.495	1304.626	1541.498	1798.108	2074.457	2370.545
λ_k^+	714.207	891.860	1089.252	1306.384	1543.253	1799.863	2076.212	2372.300
$\lambda_k^+ - \lambda_k^-$	1.756	1.756	1.757	1.758	1.755	1.755	1.755	1.755

3. Transition to a system of ordinary differential equations

Since the eigenfunctions of the operator A form a complete orthonormal system, we can expand the solution of Eq. (1.6) and the right-hand side of this equation in the system $(\mathbf{h}_j(x))_{j=1}^{\infty}$

$$\boldsymbol{\omega}(x,t) = \sum_{j=1}^{\infty} s_J(t)\mathbf{h}_j(x), \quad \mathbf{g}(x) = \sum_{i=1}^{\infty} C_j\mathbf{h}_j(x);$$
$$s_j(t) = \langle \boldsymbol{\omega}(x,t), \mathbf{h}_j(x)\rangle_H, \quad C_j = \langle \mathbf{g}(x), \mathbf{h}_j(x)\rangle_H$$

If $C_j = 0$ for a certain j, we shall say that the value of the radius of the disc r is singular. It has been proved in Ref. 5 that the set of singular values of r is not greater than a denumerable set. We shall henceforth assume that the value of r is not singular, that is, $C_j \neq 0$ for any $j \in \mathbb{N}$. After substitution into Eq. (1.6), we obtain

$$\sum_{j=1}^{\infty} [\ddot{s}_j(t) + \lambda_j s_j(t) - C_j u(t)]\mathbf{h}_j(x) = 0$$

By virtue of the orthogonality of the system of eigenfunctions, we obtain a denumerable system of ordinary differential equations

$$\ddot{s}_j(t) + \lambda_j s_j(t) = C_j u(t), \quad j \in \mathbb{N}$$

Using Parseval's equality we conclude that the functional (1.4) can be written in the form

$$\int_0^{\infty}\left(\sum_{j=1}^{\infty} s_j^2(t)\right)dt$$

Moreover, expanding the initial conditions (1.3) in a system of eigenfunctions

$$\begin{pmatrix} w_0(x) \\ \xi_0(x) \end{pmatrix} = \sum_{j=1}^{\infty} \tau_j^0\mathbf{h}_j(x), \quad \begin{pmatrix} w_1(x) \\ \xi_1(x) \end{pmatrix} = \sum_{i=1}^{\infty} \tau_j\mathbf{h}_j(x)$$

we obtain the initial conditions for the functions $s_j(t)$: $s_j(0) = \tau_j^0$, $s'_j(0) = \tau$.

For the majority of initial states, associated with natural external actions on the beam, the main part of the vibrational energy occurs in the fundamental normal mode and, as a first approximation when constructing the optimal solution in problem (1.1)–(1.4), it therefore makes sense to consider the problem of the optimal quenching of the first mode. The criterion of optimality, that is, the minimization of the mean square deviation facilitates minimal action on the remaining vibration modes during the process of the quenching the principal mode.

In this way, we arrive at the following optimal control problem, that is, the problem of a controllable harmonic oscillator with a quadratic cost functional: it is required to minimize the functional

$$\int_0^{\infty} s^2(t)dt \tag{3.1}$$

in the trajectories of the control system

$$\ddot{s}(t) + \lambda s(t) = Cu(t) \tag{3.2}$$

where s is a phase variable, u is a scalar control, $-1 \leq u \leq 1$ and the initial conditions $s(0) = \tau_1^0$, $\dot{s}(0) = \tau_1$. The structure of the optimal control, of the switching curve and the properties of the optimal trajectories for this problem have been investigated earlier in Refs. 7,8.

4. Specific modes and modes with chattering

The construction of the optimal synthesis for problem (3.1), (3.2) is based on a technique developed in a number of papers.[7,9] We will present a brief review of the corresponding results and consider the problem of minimizing the functional

$$\int_0^T (\varphi_0(x) + u\varphi_1(x))dt$$

on the set of solutions of the control system

$$\dot{x} = f_0(x) + uf_1(x)$$

with the boundary conditions $x(0) \in B_0 \subset \mathbb{R}^n$, $x(T) \in B_T \subset \mathbb{R}^n$. Here x is the phase coordinate, u is a scalar control and $u \leq 1$. The mappings $\varphi_i : \mathbb{R}^n \to \mathbb{R}$, $f_i : \mathbb{R}^n \to \mathbb{R}(i = 0, 1)$ are specified by functions of the class C^∞ and B_0, B_T are smooth manifolds. The admissible controls $u(t)$ are measurable and the corresponding trajectories $x(t)$ are absolutely continuous. In order to solve the problem, the Hamiltonian $H = H_0(x, \psi) + uH_1(x, \psi)$ is set up, where

$$H_0(x, \psi) = f_0(x)\psi - \frac{1}{2}\varphi_0(x), \quad H_1(x, \psi) = f_1(x)\psi - \frac{1}{2}\varphi_1(x)$$

Using Pontryagin's maximum principle, we construct the Hamilton system

$$\dot{x} = \frac{\partial H}{\partial \psi}, \quad \dot{\psi} = -\frac{\partial H}{\partial x} \tag{4.1}$$

with Hamiltonian H. The maximum condition determines the optimal control: $u = +1$ when $H_1 > 0$ and $u = -1$ when $H_1 < 0$: $H_1 = 0$ is the surface of discontinuity of the Hamilton system.

The extremal $x(t)$, $\psi(t)$ of Pontryagin's maximum principle is said to be singular on a segment (t_0, t_1) if $H_1(x(t), \psi(t)) = 0$ when $t \in (t_0, t_1)$. This means that the extremal lies on the surface $H_1 = 0$. In order to find the control on a sigular trajectory, it is necessary to differentiate the identity with respect to t by virtue of system (4.1) until a control u with a non-zero coefficient occurs in the resulting expression. It is well-known that this can only take place in an even step 2q of differentiation. The number q is called the order of a singular trajectory. We shall say that the intrinsic order of a singular trajectory is equal to q if

$$ad_{H_1} ad_{H_0}^i H_1 = 0, \quad i = 0, \ldots, 2q - 2, \quad ad_{H_1} ad_{H_0}^{2q-1} H_1 \neq 0$$

in some open neighbourhood of a singular trajectory. If the equalities are only valid at points of the extremal $(x(t), \psi(t)$, $t \in (t_0, t_1)$, we shall say that q is the local order of the extremal.

Suppose the intrinsic order of a singular trajectory is equal to q. If the functions

$$z_{i+1}(x, \psi) = ad_{H_0}^i H_1, \quad i = 0, \ldots, 2q - 1$$

are independent in the neighbourhood of the extremal, then, on supplementing them up to an independent system with the functions $w_j(x, \psi)(j = 1, \ldots, 2n - 2q)$ and adopting (z, w) as the new variables, the Hamilton system for Pontryagin's maximum principle can be reduced to the form

$$\dot{z}_i = z_{i+1}, \quad i = 1, \ldots, 2q - 1, \quad \dot{z}_{2q} = \rho_1(z, w) + u\rho_2(z, w)$$
$$\dot{w}_j = F_j(z, w, u), \quad j = 1, \ldots, 2n - 2q, \quad u = \operatorname{sgn} z_1$$

The singular states lie on the manifold $z = 0$. Chattering arises on joining up the non-singular states with singular states. By the term chattering trajectory, we mean a trajectory which has an infinite number of switchings of the control in a finite time interval.

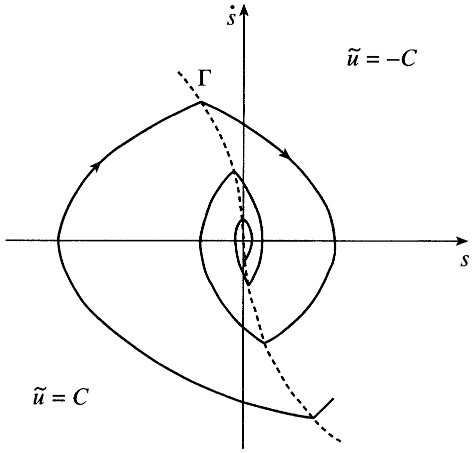

Fig. 2.

5. Optimal synthesis for a controlled harmonic oscillator with a quadratic cost functional

We put

$$\mathbf{S}(t) = \begin{pmatrix} s(t) \\ \dot{s}(t) \end{pmatrix}, \quad \mathbf{S}_0 = \begin{pmatrix} \tau_1^0 \\ \tau_1 \end{pmatrix}.$$

The following theorem arises from results obtained previously[7,10] for problem (3.1), (3.2).

Theorem 3.

1 The origin of coordinates is a singular trajectory of orders 2.
2 A neighbourhood of the origin of coordinates U_ε exists such that:
  a) *for any $\mathbf{S}_0 \in U_\varepsilon$ an optimal trajectory $\mathbf{S}^*(t)$ exists with initial condition $\mathbf{S}^*(0) = \mathbf{S}_0$, which reaches the origin of coordinates in a finite time with an infinite number of control switchings;*
  b) *in the neighbourhood U_ε the switching curve has the form*

$$\Gamma = \begin{cases} s = \mu_1(\dot{s})\dot{s}^2, & \dot{s} > 0 \\ s = \mu_2(\dot{s})\dot{s}^2, & \dot{s} < 0 \end{cases}, \quad \mu_i(\dot{s}) \in C^1, \quad \mu_1(0) \in \left(-\frac{1}{2C}, 0\right), \quad \mu_2(0) \in \left(0, \frac{1}{2C}\right)$$

The optimal control $\tilde{u}(s, \dot{s}) = -C$ above the curve Γ and $\tilde{u}(s, \dot{s}) = C$ under the curve Γ (Fig. 2).

6. Construction of a sub-optimal mode for a controlled harmonic oscillator with a quadratic cost functional

The optimal trajectories of a controlled harmonic oscillator with a quadratic cost functional have an infinite number of switchings of the control in a finite time interval. The need arises to approximate the optimal chattering trajectories with trajectories with a finite number of control switchings. In order to construct a suboptimal trajectory with a finite number of switchings, we will consider two auxiliary problems: the problem of the fastest stopping of the oscillator and the simple time-optimal problem.

6.1. Problem of the fastest stopping of an oscillator

$T \to \min$

$$\ddot{s}(t) + \lambda s(t) = u(t), \quad \mathbf{S}(0) = \mathbf{S}_0, \quad \mathbf{S}(T) = 0$$

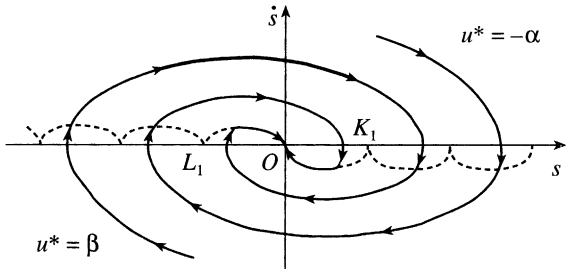

Fig. 3.

The scalar control parameter u varies in the interval $[-\alpha, \beta]$, where $\alpha>0$, $\beta>0$. An optimal synthesis was constructed in Ref. 11 for the case when $\alpha = \beta = 1$. The construction is easily extended to the case of arbitrary α and β. The optimal synthesis is constructed in the following manner Fig. 3: when $u = \beta$, the phase trajectories of the system are ellipses $(\sqrt{\lambda}s - \beta/\sqrt{\lambda})^2 + \dot{s}^2 = d^2$ with their centre at the point $(\beta/\sqrt{\lambda}, 0)$ and, when $u = -\alpha$, the ellipses $(\sqrt{\lambda}s + \alpha/\sqrt{\lambda})^2 + \dot{s}^2 = d^2$ with centre at the point $(-\alpha/\sqrt{\lambda}, 0)$. A phase point moves clockwise along the ellipses. Switchings of the control occur on a separate curve which consists of a denumerable number of semi-ellipses. Above the switching curve and in the arc OL_1, the optimal control is $u^* = -\alpha$ and, below the switching curve and in the arc OK_1, the optimal control is $u^* = \beta$. Any optimal trajectory has a finite number of switching points which depends on the initial conditions. The further the initial point from the origin of coordinates, the greater the number of control switching points.

The optimal synthesis in the problem of the fastest stopping of an oscillator is quite lengthy. However, for the stopping of an oscillator, an optimal control for the time-optimal problem, which is simple in its implementation, can be used, and the system reaches the origin of the coordinates in a finite time. An estimate of the error in the time of reaching the origin of the coordinates compared with the optimal solution has been presented in Ref. 10.

6.2. The time-optimal problem

$$T \to \min$$

$$\ddot{s}(t) = u(t), \quad S(0) = S_0, \quad S(T) = 0$$

The scalar control is bounded: $-\alpha \le u \le \beta$, $\alpha > 0$, $\beta > 0$. It is well-known[11] that the optimal control $\hat{u}(t)$ is the bang-bang control and it has at most one switching.

We will now describe the process of constructing a suboptimal solution for problem (3.1), (3.2). For the point $\mathbf{S}_0$, we consider a segment of the trajectory $\mathbf{S}^*(t)$ ($\mathbf{S}^*(0) = \mathbf{S}_0$), which is optimal in problem (3.1), (3.2), containing exactly N switchings. Suppose this segment ends at the point $\mathbf{S}_N$. Starting from the point $\mathbf{S}_N$, we use the control $\hat{u}(\mathbf{S}_N)$ or $u^*(\mathbf{S}_N)$ ($\hat{u}(\mathbf{S}_N)$ is the optimal control in the time-optimal problem and $u^*(\mathbf{S}_N)$ is the optimal control in the problem of the stopping of the oscillator). Suppose $\hat{J}_N(\mathbf{S}_0)$ is the value of the functional on the trajectory which has been constructed in this way and $J^*(\mathbf{S}_0)$ is the value of the functional on the optimal trajectory $\mathbf{S}^*(t)$.

Theorem 4. *An ε-neighbourhood of the origin O_ε and constants $M > 0$, $\nu > 0$ exist such that following the estimate holds for any point $\mathbf{S}_0 \in O_\varepsilon$*

$$\hat{J}_N(\mathbf{S}_0) - J^*(\mathbf{S}_0) < Me^{-\nu N}$$

Hence, in practice, a control with a finite number of switchings in a finite interval of the motion can be used; at the same time, it is possible to ensure that the error in the functional is as small as desired.[12]

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